§4.6 Quantum Chromodynamics

Let's compute the β-function of QCD.
For this we use the following method:

Minimal Subtraction:

Bare coupling g_B has poles as a function of space-time dimensionality d

$\longrightarrow$ residues fixed by requiring g_R to be regular as $d \to 4$

Suppose $[g_B] = \Lambda^{\Delta(d)}$, for some $\Delta(d)$, e.g. for non-abelian gauge theories

$$\left[\tfrac{1}{4} g_B^{-2} F^\alpha_{\mu\nu} F^{\alpha\mu\nu}\right] = \Lambda^d$$

$$\Longrightarrow [g_B] = \Lambda^{(4-d)/2} \longrightarrow \Delta(d) = \frac{4-d}{2}$$

Thus we get

$$\underbrace{g_B(d)\, \kappa^{-\Delta(d)}}_{\text{dimensionless}} = g(\kappa, d) + \sum_{\nu=1}^{\infty} \varepsilon^{-\nu}\, b_\nu(g(\kappa,d)) \quad (1)$$

where $\varepsilon = d-4$

coming from divergent loop amplitudes in $d=4$

Now differentiate eq. (1) by κ :

$$\rightarrow \kappa \frac{\partial}{\partial \kappa}\left(g_B \kappa^{-\Delta(d)}\right) = -\Delta(d)\, g_B \kappa^{-\Delta(d)}$$

$$= \underbrace{\kappa \frac{\partial}{\partial \kappa} g(\kappa, d)}_{=\beta(g,d)} + \sum_{\nu=1}^{\infty} b'_\nu(g) \beta(g,d) \varepsilon^{-\nu} \quad (2)$$

where $b'_\nu(g) \equiv \frac{\partial}{\partial g} b_\nu(g)$

Write

$$\Delta(d) = \Delta + c(d-4)$$

$\rightarrow$ left-hand side of (2) becomes

$$-cg\varepsilon - \left[\Delta g + b_1(g) c\right] - \sum_{\nu=1}^{\infty} \varepsilon^{-\nu} \left[c b_{\nu+1}(g) + \Delta b_\nu(g)\right] \quad (3)$$

$\rightarrow$ highest power of ε is 1, so the same must be true for the rhs of (2):

$$\beta(g,d) = \beta(g) + \alpha(g)\varepsilon$$

(no negative powers of ε as β is a finite function at $\varepsilon = 0$)

equating terms on lhs and rhs of (2) gives:

$$\alpha(g) = -cg$$

Inserting back into (2) gives:

$$\text{rhs} = \beta(g) - cg\varepsilon + \sum_{\nu=1}^{\infty} b_\nu'(g)\big(\beta(g) - cg\varepsilon\big)\varepsilon^{-\nu}$$

$$= -cg\varepsilon + \beta(g) - b_1'(g)\,cg$$

$$+ \sum_{\nu=1}^{\infty}\Big(b_\nu'(g)\beta(g) - b_{\nu+1}'(g)cg\Big)\varepsilon^{-1} \qquad (4)$$

Comparing with (3) gives:

$$\beta(g) = -\Delta g - b_1(g)c + b_1'(g)cg \qquad (5)$$

Equating polar terms in ε gives the following recursion relation:

$$c b_{\nu+1}(g) - cg\, b_{\nu+1}'(g) = -\Delta b_\nu(g) - b_\nu'(g)\beta(g)$$

$\longrightarrow$ $b_{\nu+1}$ is determined from b_ν and hence from b_1

Specifying to non-abelian gauge theories, we have $c = -\frac{1}{2}$ and $\Delta = 0$, thus:

$$\beta(g) = \frac{1}{2}\left[b_1(g) - g b_1'(g)\right] \qquad (6)$$

Now we apply our result to QCD:

QCD is non-Abelian gauge theory with

- gauge group: $SU(3)$
- matter content: spin $\frac{1}{2}$ particles

  known as "quarks"

  u, c, and quarks with $U(1)_e$ charge $2e/3$, and d, s, and b quarks with $U(1)_e$ charge $-e/3$

  $\to$ 6 "flavors"

  quarks of each flavor come in 3 "colors"

  $\to$ fundamental representation of $SU(3)$

- Baryons (protons, neutrons) are color-neutral bound states of 3 quarks (using anti-sym tensor $p = \varepsilon_{ijk} u^i d^j d^k$)
- Mesons are color-neutral bound states of quarks and anti-quarks.

$\to$ renormalized gauge coupling is given by $g_R = g_B \left[1 + \frac{g_B^2}{4\pi^2} \ln\left(\frac{\Lambda}{k}\right)\left(\frac{11}{12} C_1 - \frac{1}{3} C_2\right) + O(g_B^4)\right]$

Recall that the term $\ln\left(\frac{\Lambda}{k}\right)$ comes from

$$X = 2\pi^2 i \int_k^\Lambda \frac{dq}{q} = 2\pi^2 i \ln\left(\frac{\Lambda}{k}\right)$$

Alternatively, we can evaluate x by writing

$$X = i \int_0^\infty \frac{2\pi^2 q^{d-1} dq}{(q^2+k^2)^2}$$

where d is allowed to approach 4 at the end and k is infrared cut-off:

evaluate $\frac{\delta^4 \Gamma[A]}{\delta A^4}$ not at $A=0$

but instead compute

$$\left.\frac{\delta^4 \Gamma[A]}{\delta A^4}\right|_{gA=k}$$

$\longrightarrow$ propagators effectively become

$$\frac{1}{q^2+k^2}$$

Analytically continuing d to $\mathbb{C}$, we get

$$X = -i\pi^2\left(\frac{d}{2}-1\right) k^{d-4}\, \pi \Big/ \sin\left(\left(\frac{d}{2}-2\right)\pi\right)$$

$$= -2i\pi^2 \left[\frac{1}{d-4} + \ln k + \cdots\right],$$

thus we have

$$g_B = g_R + \frac{g_R^3}{4\pi^2}\left(\frac{11}{12}C_1 - \frac{1}{3}C_2\right)\left(\frac{1}{d-4} + \ln K + \cdots\right)$$
$$+ O(g_R^5) \qquad (7)$$

where

$$C_{\gamma\alpha\beta}\, C_{\delta\alpha\beta} = g^2 C_1\, \delta_{\gamma\delta}$$
$$\mathrm{tr}\{t_\gamma t_\delta\} = g^2 C_2\, \delta_{\gamma\delta}$$

From (7) we get

$$b_1(g) = \frac{g^3}{4\pi^2}\left(\frac{11}{12}C_1 - \frac{1}{3}C_2\right) + O(g^5)$$

and from

$$\beta(g) = \frac{1}{2}\left[b_1(g) - g b_1'(g)\right]$$

we then have

$$\beta(g) = -\frac{g^3}{4\pi^2}\left(\frac{11}{12}C_1 - \frac{1}{3}C_2\right) + O(g^5) \qquad (8)$$

For SU(3) theory with n_f quarks:

$$C_1 = 3, \qquad C_2 = n_f/2$$

$$(8) \longrightarrow \beta(g) = -\frac{g^3}{4\pi^2}\left(\frac{11}{4} - \frac{1}{6}n_f\right) + O(g^5)$$

For $n_f \leq 16$ theory is asymptotically free!

Now let us solve the equation

$$k \frac{d}{dk} g(k) = -\frac{g^3(k)}{4\pi^2}\left(\frac{11}{4} - \frac{1}{6}n_f\right)$$

$$\longrightarrow \alpha_s(k) \equiv \frac{g^2(k)}{4\pi} = \frac{12\pi}{(33-2n_f)\ln(k^2/\Lambda^2)}$$

<u>Check</u>:

$$k\frac{d}{dk} g^2 = k\, 2g \frac{d}{dk} g$$

$$= -\frac{g^4}{2\pi^2}\left(\frac{11}{4} - \frac{1}{6}n_f\right)$$

and

$$k\frac{d}{dk}\alpha_s(k) = \frac{12\pi}{(33-2n_f)}\left(-\frac{2}{\ln(k^2/\Lambda^2)^2}\right)$$

$$= -2\pi \frac{1}{\left(\frac{11}{4} - \frac{1}{6}n_f\right)\ln(k^2/\Lambda^2)^2}$$

$$= -\frac{\left(\frac{11}{4} - \frac{1}{6}n_f\right)}{2\pi^2} \underbrace{\frac{4\pi^3}{\left[\left(\frac{11}{4} - \frac{1}{6}n_f\right)\ln(k^2/\Lambda^2)\right]^2}}_{= \frac{g^4}{4\pi}} \checkmark$$